

# Blackbody Radiation and the Scaling Symmetry of Relativistic Classical Electron Theory with Classical Electromagnetic Zero-Point Radiation

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## Abstract

It is pointed out that relativistic classical electron theory with classical electromagnetic zero-point radiation has a scaling symmetry which is suitable for understanding the equilibrium behavior of classical thermal radiation at a spectrum other than the Rayleigh-Jeans spectrum. In relativistic classical electron theory, the masses of the particles are the only scale-giving parameters associated with mechanics while the action-angle variables are scale invariant. The theory thus separates the interaction of the action variables of matter and radiation from the scale-giving parameters. Classical zero-point radiation is invariant under scattering by the charged particles of relativistic classical electron theory. The basic ideas of the matter-radiation interaction are illustrated in a simple relativistic classical electromagnetic example.

## I. INTRODUCTION

Although scaling symmetry receives very little attention within classical physics, an appreciation of this symmetry is crucial for understanding blackbody radiation within classical theory. Traditional classical electron theory with its arbitrary nonrelativistic potentials is presented as though the scales of length, time, and energy can all be chosen independently. However, this is not the scaling symmetry which appears in nature. Nature has chosen a  $\sigma_{ltU^{-1}}$ -scaling which links together the scales of length, time, and energy. The links between the scales are presented by several fundamental constants, including the speed of light  $c$  connecting length  $l$  and time  $t$  ( $c = l/t$ ) and the elementary charge  $e$  connecting energy  $U$  and length  $l$  ( $U = e^2/l$ ). Here we point out that the restrictions within  $\sigma_{ltU^{-1}}$ -scaling give relativistic classical electron theory with classical zero-point radiation an additional symmetry which gives stability to the zero-point radiation spectrum and allows the possibility of a universal equilibrium spectrum for classical thermal radiation.

Relativistic classical electron theory with classical zero-point radiation consists of Newton's second law for the motion of particles, all of the same fixed charge  $e$  and with various masses  $m$ , under the action of the Lorentz force due to electromagnetic fields described by Maxwell's equations, with the homogeneous boundary condition on Maxwell's equations corresponding to random classical electromagnetic radiation with a Lorentz-invariant spectrum, classical electromagnetic zero-point radiation.[1][2][3] The one factor setting the scale of the classical zero-point radiation is chosen so as to provide agreement with the experimentally observed Casimir forces and is recognizable as  $(1/2)\hbar$  where  $\hbar$  is Planck's constant  $h$  divided by  $2\pi$ . Here the multiplicative constant  $\hbar$  of the classical zero-point radiation is another fundamental constant which links energy and time. The three fundamental constants  $e$ ,  $c$ , and  $\hbar$  allow the formation of one pure number with no units  $e^2/\hbar c$ , known as the fine-structure constant. Classical electron theory with classical zero-point radiation has provided calculations in agreement with experiment for a number of phenomena which are usually regarded as lying outside the domain of classical physics, including the Planck spectrum of blackbody radiation, specific heats of solids, diamagnetism, Casimir forces, van der Waals forces,[4] and the ground state of hydrogen.[5]

Although the Planck spectrum for blackbody radiation has been derived from a number of lines of reasoning using classical physics with classical zero-point radiation,[6] the

problem of classical radiation equilibrium has never been completely solved within classical physics. Classical thermal radiation equilibrium requires that the spectrum of thermal radiation be stable under scattering by a "black particle." Previous scattering calculations for nonrelativistic mechanical scatterers have all produced the Rayleigh-Jeans spectrum as the equilibrium spectrum.[7][8][9][10] Indeed, these nonrelativistic scatterers act to transform the zero-point radiation spectrum towards the Rayleigh-Jeans spectrum.[8] However, all of the previous scattering calculations violate the  $\sigma_{ltU^{-1}}$ -scaling behavior of relativistic classical electron theory. It turns out that the zero-point radiation spectrum is invariant under scattering by a relativistic classical hydrogen atom. Specific  $\sigma_{ltU^{-1}}$ -scaling behavior is needed for the universal character of the blackbody spectrum.

The outline of this article is as follows. In the first part we discuss what is meant by scaling symmetry and how it is related to the interaction of radiation and matter. We note that only the Coulomb potential appears in relativistic classical electron theory and show that it allows a separation of the interaction of the  $\sigma_{ltU^{-1}}$ -scale-invariant action variables of both matter and radiation from the  $\sigma_{ltU^{-1}}$ -scale-giving parameters of mass and frequency. We then note that zero-point radiation is invariant under scattering by a classical hydrogen atom. The second part of the article is devoted to a simple example of a charged particle held in a circular Coulomb orbit by a circularly-polarized plane wave. The example illustrates explicitly the separation of the behavior of the  $\sigma_{ltU^{-1}}$ -scale-invariant parameters from the scaling parameters in the interaction of relativistic matter and radiation. The example also suggests how the thermal radiation spectrum can take a universal form. Finally we end with remarks on the changes in classical statistical mechanics and in classical electron theory which are involved in our understanding of nature within classical physics.

## II. PART I - $\sigma_{ltU^{-1}}$ -SCALING SYMMETRY AND THE INTERACTION OF MATTER AND RADIATION

### A. Scaling Symmetry

A set is said to be "invariant" under a scale change if the set is mapped onto itself under the scaling operation. A  $\sigma_{ltU^{-1}}$ -scale change simultaneously maps lengths as  $l \rightarrow l' = \sigma_{ltU^{-1}}l$ , maps times as  $t \rightarrow t' = \sigma_{ltU^{-1}}t$ , and maps energies as  $U \rightarrow U' = U/\sigma_{ltU^{-1}}$  where  $\sigma_{ltU^{-1}}$  is

chosen as some positive real number.[11] Such a scaling operation may be regarded as a change in the units of measurement, but necessarily a simultaneous change of all three fundamental units. For an example of  $\sigma_{lU^{-1}}$ -scale invariance, consider the classical zero-point radiation spectrum given by the set of all normal modes with a gaussian distribution of electric field amplitudes and average energy per normal mode satisfying the relationship  $U_\omega = (1/2)\hbar\omega$ . Under a  $\sigma_{lU^{-1}}$ -scale change, the frequency  $\omega$  (with units of inverse time) of a normal mode is mapped to  $\omega' = \omega/\sigma_{lU^{-1}}$  while the energy  $U_\omega$  is mapped to  $U'_{\omega'} = U_\omega/\sigma_{lU^{-1}}$ . But then the functional relationship defining zero-point radiation is unchanged since  $U'_{\omega'} = U_\omega/\sigma_{lU^{-1}} = (1/2)\hbar\omega/\sigma_{lU^{-1}} = (1/2)\hbar\omega'$ . Thus this distribution is mapped onto itself, and we say that the zero-point energy spectrum is  $\sigma_{lU^{-1}}$ -scale invariant. On the other hand, the thermal radiation spectrum of all the normal modes at a single temperature  $T$  is not  $\sigma_{lU^{-1}}$ -scale invariant because the spectrum depends on the temperature  $T$  (with units related to energy) and is mapped onto the spectrum depending upon temperature  $T' = T/\sigma_{lU^{-1}}$ . However, the one-parameter *collection* labeled by  $T$  of all thermal radiation spectra is indeed  $\sigma_{lU^{-1}}$ -scale invariant because the *collection* is mapped onto itself.

An individual mass  $m$  is not  $\sigma_{lU^{-1}}$ -scale invariant since under scaling  $m$  (with units related to energy) is mapped to  $m' = m/\sigma_{lU^{-1}}$ . However, the one-parameter *collection* labeled by  $m$  of all masses is  $\sigma_{lU^{-1}}$ -scale invariant. The individual constants  $e$ ,  $c$ , and  $\hbar$  are all  $\sigma_{lU^{-1}}$ -scale invariant because they have dimensions such that the factors of  $\sigma_{lU^{-1}}$  appearing in a simultaneous scale change cancel completely. We speak of a "scaling variable" or a "scaling parameter" as one which changes under the action of a  $\sigma_{lU^{-1}}$ -scale change. Thus for example, the mass  $m$  of a particle, the frequency  $\omega$  of a normal mode, and the temperature  $T$  of a system at equilibrium are all scaling parameters. On the other hand, the charge of a particle and its angular momentum are not scaling parameters since they are  $\sigma_{lU^{-1}}$ -scale invariant.

## B. $\sigma_{lU^{-1}}$ -Scaling Symmetry and Adiabatic Compression

Classical electromagnetic theory contains no fundamental length. Accordingly, for pure radiation in an spherical enclosure, a  $\sigma_{lU^{-1}}$ -scale change is indistinguishable from a spherical adiabatic compression; the normal mode frequencies, the normal mode energies, and the volume all change in the same way.[11] However, if the enclosure contains radiation and

also a mass  $m$ , then a change of scale is very different from an adiabatic compression of the enclosure. Under a  $\sigma_{tU}^{-1}$ -scale change, the radiation and the mass will all be mapped to new values; however, under an adiabatic compression, the scales of radiation energy, frequency, and volume will all change while leaving the mass  $m$  of the particle unchanged. The great fascination of blackbody radiation during the nineteenth century was its universal character. How could the radiation spectrum which corresponded to thermal equilibrium be independent of the mass  $m$  of a charged particle which scattered the radiation? How could the *form* of the equilibrium radiation spectrum in the enclosure be unchanged after an adiabatic compression which altered the ratios of the parameters of the radiation to the particle mass  $m$  in the enclosure?

Modern physics has provided an answer to this question by changing the rules of interaction away from classical physics and over to quantum theory. However, we wish to point out that a solution within classical physics consists in simply insisting on relativistic theory with fixed charge  $e$  rather than allowing the mixtures of relativistic and nonrelativistic systems which have appeared in previous classical analyses. Many physicists do not seem to be aware of the no-interaction theorem of Currie, Jordan, and Sudarshan,[12] and the fact that special relativity imposes stringent restrictions on the interactions between particles.[13] Relativistic classical electron theory requires that the particles interact through electromagnetic fields. The particles can not interact through potentials other than the Coulomb potential which arises from electromagnetic fields. Indeed relativistic classical electron theory with classical electromagnetic zero-point radiation has particle masses and normal mode frequencies as essentially the only scaling parameters, and therefore the theory has a  $\sigma_{tU}^{-1}$ -scaling behavior which is different from any theory which allows arbitrary interaction potentials. The  $\sigma_{tU}^{-1}$ -scaling behavior provides the additional symmetry which decouples the action variables from the scale-giving variables. This decoupling is precisely what is wanted for a universal character of thermal radiation. In the next section, we will discuss this decoupling, then in Part II we will give a specific model example of the connection between a relativistic scatterer and radiation.

### C. Action-Angle Variables for Systems

The action-angle variables  $J, \theta$  of oscillating systems have dimensions which make them invariant under  $\sigma_{lU^{-1}}$ -scale changes. Thus, for example, the angular momentum of a particle of mass  $m$  is an action variable having dimensions of *mass*  $\times$  *velocity*  $\times$  *length*. Since mass transforms as an inverse length while the velocity is invariant under  $\sigma_{lU^{-1}}$ -scale changes, the angular momentum is indeed  $\sigma_{lU^{-1}}$ -scale invariant.

Each of the normal modes of radiation oscillation in a cavity can be regarded as an independent harmonic oscillator system.[14] When expressed in terms of action-angle variables, the energy  $U_\omega$  of a mode is related to the frequency  $\omega$  of the mode and the action variable  $J_\omega$  as

$$U_\omega = J_\omega \omega \quad (1)$$

But then the ratio  $U_\omega/\omega$  is given by

$$U_\omega/\omega = J_\omega \quad (2)$$

and is  $\sigma_{lU^{-1}}$ -scale invariant, just as the action variable  $J_\omega$  is  $\sigma_{lU^{-1}}$ -scale invariant. Thus for the radiation mode, we find here that the ratio between the scale-giving parameter  $\omega$  and the energy  $U_\omega$  depends solely on the  $\sigma_{lU^{-1}}$ -scale-invariant action variable  $J_\omega$ . Under an adiabatic compression, the frequency  $\omega$  of the mode will change and the energy  $U_\omega$  will change, but the action variable  $J_\omega$  is an adiabatic invariant and will not change. Thus a  $\sigma_{lU^{-1}}$ -scale change is the same as an adiabatic compression of the mode. The action variable of the radiation is not changed in magnitude in either case. For this system there is a complete decoupling of the ratio of the scale-giving parameter  $\omega$  and the energy  $U_\omega$  from anything but the action variable  $J_\omega$ .

In relativistic classical electron theory with classical electromagnetic zero-point radiation, the charges interact through the electromagnetic fields. In the mechanical approximation which excludes radiation, the particles of relativistic classical electron theory can be regarded as interacting through the Coulomb potential. As an example of this situation, we consider a particle of charge  $e$  and mass  $m$  in the Coulomb field of another particle of charge  $-e$  and very large mass, corresponding to a classical hydrogen atom. The energy of the system

(excluding the self-energy of the large mass) is given by[15]

$$U_m = m\gamma c^2 - e^2/r$$

$$= mc^2 \left( 1 + \frac{(e^2/c)^2}{\{J_3 - J_2 + [J_2^2 - (e^2/c)^2]^{1/2}\}^2} \right)^{-1} \quad (3)$$

where  $\gamma = (1 - v^2/c^2)^{-1/2}$ , while  $J_2$  and  $J_3$  are the action variables for the hydrogen system. If we divide through by  $mc^2$  to obtain  $U_m/(mc^2)$ , then this ratio is equal to a function of the action variables (and the fixed constant  $e^2/c$ ) alone, and is not dependent on any scale-giving parameter. Once again we have a decoupling of the ratio of the system's scale-giving parameter  $m$  and the energy  $U$  from anything but the action variables  $J_i$ . Indeed, all lengths, times, and energies for the relativistic hydrogen orbits will involve respectively the fundamental length  $e^2/(mc^2)$  times a function of the  $J$ 's, the fundamental time  $e^2/(mc^3)$  times a function of the  $J$ 's, and the fundamental energy  $mc^2$  times a function of the  $J$ 's. The orbital speed of the mass  $m$  is invariant under  $\sigma_{hU^{-1}}$ -scale change and will involve the  $J$ 's only.

These relations can be seen easily for the restricted case of a circular orbit where Newton's second law gives

$$m\gamma \frac{v^2}{r} = \frac{e^2}{r^2} \quad (4)$$

while the angular momentum  $J$  is

$$J = m\gamma vr \quad (5)$$

Then combining equations (4) and (5), the speed of the particle in its orbit is

$$v = \frac{e^2}{J} \quad (6)$$

the radius is

$$r = \frac{J}{m\gamma v} = \left( \frac{e^2}{mc^2} \right) \left( \frac{Jc}{e^2} \right)^2 \left[ 1 - \left( \frac{e^2}{Jc} \right)^2 \right]^{1/2} \quad (7)$$

the frequency is

$$\omega = \frac{v}{r} = v \frac{m\gamma v}{J} = \left( \frac{mc^3}{e^2} \right) \left( \frac{e^2}{Jc} \right)^3 \left[ 1 - \left( \frac{e^2}{Jc} \right)^2 \right]^{-1/2} \quad (8)$$

and the energy is

$$U = mc^2 \left[ 1 - \left( \frac{e^2}{Jc} \right)^2 \right]^{1/2} \quad (9)$$

In each case we see the appearance of the characteristic length  $e^2/(mc^2)$ , time  $e^2/(mc^3)$ , or energy  $mc^2$  times a function of the angular momentum  $J$ .

It should be emphasized that the decoupling of the  $\sigma_{lU^{-1}}$ -scale-invariant ratios from the scaling parameters is something which occurs only for the Coulomb potential and is not the usual situation for mechanical systems. Thus, for example, the energy of the nonlinear oscillator can be rewritten in terms of action variables as[16]

$$\begin{aligned} H &= p^2/(2m) + m\omega_0^2 x^2/2 + \Gamma x^3/3 \\ &= J\omega_0 - \frac{5\Gamma^2(\omega_0 J)^2}{12\omega_0 m^3} + O(\Gamma^3) \end{aligned} \quad (10)$$

This system has three scaling parameters  $m$ ,  $\omega_0$ , and  $\Gamma$  with the dimensions of *mass*,  $(\text{time})^{-1}$ , and *energy* $\times(\text{length})^{-3}$  respectively. The  $\sigma_{lU^{-1}}$ -scale-invariant ratio  $H/\omega_0 = J - (5\Gamma^2 J^2)/(12m^3) + O(\Gamma^3)$  is not a function of the action variable  $J$  alone but rather depends also upon both  $m$  and  $\Gamma$ .

#### **D. Invariance of Zero-Point Radiation Under Scattering by a Classical Hydrogen Atom**

Classical electromagnetic zero-point radiation consists of the radiation normal modes at all frequencies with an energy  $(1/2)\hbar\omega$  per normal mode and random phases between the radiation modes. Each mode involves the electromagnetic fields oscillating with their own initial phase, at their own frequency  $\omega$ , and with their action variable  $J_\omega$  taking the common value  $J_\omega = (1/2)\hbar$ . Classical thermal radiation at temperature  $T > 0$  has the same properties except that the action variable  $J_\omega$  no longer has a common value but rather varies with the frequency  $\omega$ ,  $J_\omega = F(\hbar\omega/(k_B T)) > (1/2)\hbar$  where  $F$  is some characteristic function. The last inequality means that each normal mode has an energy which is larger than the zero-point energy of the mode. The difference between the energy at finite temperature and the zero-point energy gives the thermal energy in a radiation mode. The sum of the thermal energies over all the modes gives the finite (for finite volume) thermal energy in the volume. The zero-point radiation spectrum where each action variable  $J_\omega$  take a common value is Lorentz-invariant,  $\sigma_{lU^{-1}}$ -scale invariant, and invariant under adiabatic compression.[3] Thermal radiation at  $T > 0$  has a preferred Lorentz frame where it is isotropic, and changes temperature  $T$  under a  $\sigma_{lU^{-1}}$ -scale change or under an adiabatic compression.



Classical electromagnetic radiation can not bring itself to thermal equilibrium. Rather it is the interaction with matter which redistributes the thermal radiation energy (above the zero-point energy) into the various normal modes so as to achieve equilibrium. Since thermal radiation equilibrium is determined by matter, it is clear that the symmetries of matter will profoundly affect radiation equilibrium. In particular, the  $\sigma_{lU^{-1}}$ -scaling behavior of relativistic classical electron theory with classical zero-point radiation will enforce quite different conditions from those allowed by scatterers which have a different scaling behavior.

If the classical hydrogen atom of the previous section is placed in classical zero-point radiation, then the radiation will interact with the orbiting particle of charge  $e$  and mass  $m$ , and so the radiation will be scattered. However, the distribution of values for all the action variables  $J$  will be determined independently of frequency  $\omega$  or of mass  $m$  since the action variables are decoupled from the scale-giving parameters. The incident zero-point radiation in free space has the action variables  $J_\omega$  take the same value  $J_\omega = (1/2)\hbar$  at each frequency  $\omega$ . Since there is only one scale-giving parameter  $m$  for the interaction between zero-point radiation and the Coulomb system, there is no  $\sigma_{lU^{-1}}$ -scale-invariant quantity which can be formed which involves  $m$ . Thus all the action variables of the entire system must be functions of  $\hbar$  and the pure number  $e^2/(\hbar c)$  which can be formed from the  $\sigma_{lU^{-1}}$ -scale-invariant quantities  $e$ ,  $\hbar$ , and  $c$ . But then the spectrum of the radiation must be  $\sigma_{lU^{-1}}$ -scale-invariant, and the only  $\sigma_{lU^{-1}}$ -scale-invariant spectrum is that of zero-point radiation. The hydrogen scattering system can not alter the spectrum of zero-point radiation.

The situation for a classical hydrogen atom scatterer discussed here is totally different from the previous scattering calculations appearing in the literature.[7][8][9][10] All of the previous scattering calculations involve non-Coulomb systems which do not have the  $\sigma_{lU^{-1}}$ -scaling symmetry of hydrogen. All the previous scatterers involve several mechanical parameters, as does, for example, the charged nonlinear oscillator given in Eq. (10) where  $m$ ,  $\omega_0$ , and  $\Gamma$  are all available parameters. Indeed, scattering calculations have been carried out using this charged nonlinear oscillator.[8] One finds that the scattered radiation depends explicitly on the parameters  $\Gamma$  and  $\omega_0$ , and the scattering pushes the zero-point radiation spectrum towards the Rayleigh-Jeans spectrum. In addition, it should be emphasized that all the previous scattering calculations involved systems which were not Lorentz invariant. The classical hydrogen atom of relativistic classical electron theory with classical electro-

magnetic zero-point radiation is a relativistic scattering system. The relativistic scatterer applied to the relativistically invariant zero-point spectrum would be expected to produce a relativistically invariant spectrum; i.e. to leave the zero-point spectrum invariant. The restrictions associated with  $\sigma_{ltU^{-1}}$ -scaling behavior show that no other solution is possible.

### III. PART II - RELATIVISTIC EXAMPLE OF THE INTERACTION OF MATTER AND RADIATION SHOWING A UNIVERSAL RADIATION SPECTRUM

#### A. A Simple Example

The ideas of the previous discussion can be illustrated by a simple example showing the relativistic interaction of matter and radiation. The calculation gives an insight into the possibilities of classical radiation equilibrium. Our model starts with a charged particle  $e$  of mass  $m$  in circular orbit with angular momentum  $J$  in a central potential  $V(r)$ , taken for convenience of calculation as  $V(r) = -k/r^n$ . Since the particle is charged, it emits radiation. We ask for the circularly polarized plane wave of minimum amplitude  $E_0$  incident perpendicular to the orbit which will keep the charged particle in its orbit by providing the energy lost to radiation.. A spectrum of radiation amplitude  $E_0$  versus frequency  $\omega$  is obtained by changing the mass  $m$  of the orbiting charged particle. (The effects of the magnetic field of the plane wave can be ignored. The magnetic Lorentz force can be cancelled either by supporting the particle on a frictionless surface or by introducing two circularly polarized plane waves propagating in opposite directions which are in phase and so have no magnetic field at the orbit of the particle.)[17]

The centripetal acceleration for the charge is provided by the force from the potential

$$m\gamma\frac{v^2}{r} = \frac{\partial V}{\partial r} = n\frac{k}{r^{n+1}} \quad (11)$$

while the angular momentum is given by

$$J = m\gamma vr \quad (12)$$

Combining these two equations (11) and (12), the particle speed  $v$  is given by the solution to

$$\left(\frac{v}{c}\right)^{2-n} \left(1 - \frac{v^2}{c^2}\right)^{(n-1)/2} = \frac{nk m^{n-1}}{c^{2-n} J^n} \quad (13)$$

The power emitted by the charged particle is given by  $P_{emitted} = (2/3)(e^2/c^3)\omega^4\gamma^4r^2$  while the power delivered to the charge by the incident circularly-polarized wave of electric field amplitude  $E_0$  (when the field is oriented parallel to the particle's velocity so as to provide maximum power) is  $P_{absorbed} = eE_0v$ . In steady state, the circularly polarized plane wave must have the same frequency  $\omega$  as the orbital motion of the mass  $m$ , and the power emitted must equal the power absorbed by the particle

$$\frac{2}{3}\frac{e^2}{c^3}\omega^4\gamma^4r^2 = eE_0v \quad (14)$$

Since the velocity  $v$  is related to the frequency  $\omega$  by  $v = \omega r$ , we find

$$E_0 = \frac{2}{3}\frac{e}{c^3}\omega^2v\gamma^4 \quad (15)$$

Now the electric field  $E_0$  has the units of  $(electric\ charge)/(length)^2$  so that  $E_0/\omega^2$  must be  $\sigma_{lU-1}$ -scale invariant. Indeed from Eq. (15), we find

$$\frac{E_0}{\omega^2} = \frac{2}{3}\frac{e}{c^3}v\gamma^4 \quad (16)$$

where that the right-hand side depends upon the  $\sigma_{lU-1}$ -scale-invariant velocity  $v$  and  $\sigma_{lU-1}$ -scale-invariant constants  $e$  and  $c$ .

In general, the  $\sigma_{lU-1}$ -scale-invariant ratio  $E_0/\omega^2$  for the incident radiation will have a complicated dependence upon the parameters of the mechanical system. As seen in Eq. (13), the speed  $v$  of the particle depends upon the quantity  $[nkm^{n-1}/(c^{2-n}J^n)]$ , and dependence upon this quantity continues into the expression (16) for  $E_0/\omega^2$ . Thus in general the  $\sigma_{lU-1}$ -scale invariant ratio  $E_0/\omega^2$  for the radiation depends upon the mechanical angular momentum  $J$ , the mechanical particle mass  $m$ , and the strength  $k$  of the mechanical potential. Accordingly, the radiation spectrum  $E_0$  versus  $\omega$  obtained by varying the mass  $m$  through all possible values while holding the angular momentum  $J$  fixed is not universal but rather depends upon the choice of the potential strength  $k$ .

We also note that for a general potential  $V(r) = -k/r^n$ , the potential strength  $k$  can not be a  $\sigma_{lU-1}$ -scale-invariant constant. The potential strength  $k = -Vr^n$  is transformed under a  $\sigma_{lU-1}$ -scale change as  $k \rightarrow k' = -V'(r')^n = -(V/\sigma_{lU-1})(\sigma_{lU-1}r)^n = (\sigma_{lU-1})^{n-1}k$ . Only for the Coulomb potential where  $n = 1$  (and where  $k = e^2$ ) is this constant unchanged under a  $\sigma_{lU-1}$ -scale change.

Indeed, for the case  $n = 1$  of the Coulomb potential, the situation simplifies enormously. In this case of  $V(r) = -e^2/r$ , the constant  $e$  is a  $\sigma_{lU-1}$ -scale-invariant constant, and the

equation for the particle orbital speed becomes  $v = e^2/J$ , corresponding to Eq. (6) above. In this case (and this case only), the particle speed does not depend upon the particle mass  $m$  for fixed particle angular momentum  $J$ . In the Coulomb potential, we have the  $\sigma_{lU-1}$ -scale-invariant ratio  $E_0/\omega^2$  of the stabilizing incident wave given by

$$\begin{aligned} \frac{E_0}{\omega^2} &= \frac{2}{3} \frac{e}{c^3} v \gamma^4 = \\ &= \frac{2}{3} \frac{e}{c^3} \frac{e^2}{J} \left[ 1 - \left( \frac{e^2}{Jc} \right)^2 \right]^{-2} \end{aligned} \quad (17)$$

so that this ratio depends only upon the angular momentum  $J$  of the orbiting charge and the fixed quantity  $e^2/c$ . Indeed any  $\sigma_{lU-1}$ -scale-invariant quantity for the radiation is dependent entirely upon the  $\sigma_{lU-1}$ -scale invariant quantity  $J$  for the mechanical system and has no dependence upon the scaling parameter  $m$ . When we form the spectrum  $E_0$  versus  $\omega$  of incident radiation versus frequency by changing the mass  $m$  of the orbiting charge, we find a unique radiation spectrum  $E_0 = \omega^2 F(J)$  where the function  $F(J)$  depends only on the mechanical  $J$  at the same frequency  $\omega$  (and on the quantity  $e^2/c$ ), but does not depend upon the mass  $m$ . For fixed charge  $e$ , there is a unique connection between the orbit of the particle labeled by  $(J, m)$  and the stabilizing electromagnetic radiation spectrum labeled by  $(E_0, \omega)$ . Indeed, if we keep  $J$  at a fixed value while changing  $m$ , we obtain a radiation spectrum where  $F(J)$  is a constant independent of  $\omega$ , which makes the spectrum  $\sigma_{lU-1}$ -scale invariant. This spectrum is the example's analogue of zero-point radiation.

## B. Radiation Emission into Harmonics

The situation of our simple example corresponds to the scattering of classical electromagnetic radiation but not to the scattering of *random* classical radiation. It is the scattering of *random* radiation which is involved with thermal equilibrium. Nevertheless, the example gives an idea of what is involved in the interaction of radiation and *relativistic* classical matter in the separation of the  $\sigma_{lU-1}$ -scale-invariant quantities from the scaling parameters. Indeed, the model also illustrates a scattering aspect of the interaction of matter and radiation. The incident radiation from the circularly polarized plane wave is partially absorbed by the orbiting charged particle; the radiation energy is then scattered into different directions and into the harmonic frequencies  $n\omega$  of the mechanical motion. Indeed, the radiation

emitted per unit solid angle into the  $n$ th harmonic by a charge  $e$  in uniform circular motion at frequency  $\omega_0$  is given by[18]

$$\frac{dP_n}{d\Omega} = \frac{e^2 \omega_0^2}{2\pi c} (n\beta)^2 \left\{ \left[ \frac{dJ_n(n\beta \sin \theta)}{d(n\beta \sin \theta)} \right]^2 + \frac{\cot^2 \theta}{\beta^2} J_n^2(n\beta \sin \theta) \right\} \quad (18)$$

where  $J_n(n\beta \sin \theta)$  is the Bessel function of order  $n$  evaluated at the argument  $n\beta \sin \theta$ . We see that the relative radiation emitted into the  $n$ th harmonic depends upon  $n\beta = nv/c$ . However, it then follows that for a charged particle  $e$  in a Coulomb orbit where  $\beta = e^2/Jc$ , the relative power emitted into the harmonics is a function of the particle action variable  $J$  alone with no dependence upon the mass  $m$  of the orbiting particle. This sort of independence from the mass  $m$  is just what is needed for stability of the zero-point radiation spectrum under scattering by a classical hydrogen atom. This sort of independence does not arise for any potential function other than the Coulomb potential.

### C. Connecting the Hydrogen Scatterer and the Coherent Radiation Spectrum When $J$ is a Function of $m/T$

In the treatment of our simple matter-radiation example thus far, we have emphasized the case of fixed particle angular momentum  $J$  independent of the mass  $m$ , and we showed that the needed incident circularly-polarized plane wave perpendicular to the orbit of minimum amplitude corresponded to a  $\sigma_{hU-1}$ -scale-invariant spectrum of electromagnetic waves. Now we wish to go beyond this scale-invariant spectrum. Even if we choose the particle angular momentum  $J$  not as a constant (mass-independent) value but rather choose  $J$  as a function of  $m/T$  for some constant  $T$ , we can still obtain a unique incident radiation spectrum  $E_0$  versus  $\omega$  where the  $\sigma_{hU-1}$ -scale-invariant ratio  $E_0/\omega^2$  is a functions of  $\omega/T$  where  $\omega$  is the frequency of the associated wave. This is exactly the sort of association which we expect at equilibrium for random thermal radiation.

Again the crucial aspect is the decoupling of the  $\sigma_{hU-1}$ -scale-invariant quantities from the scale-carrying quantities  $m$  for the matter and  $\omega$  for the radiation. The frequency  $\omega$  of the particle orbit (and also of the associated circularly-polarized plane wave in our example) is given in Eq. (8) while the  $\sigma_{hU-1}$ -scale-invariant ratio  $E_0/\omega^2$  of the associated circularly polarized plane wave is given in Eq. (17). The particle angular momentum  $J$  is now regarded as a function of  $m/T$ . Then the frequency  $\omega$  of the motion given in Eq. (8)

can be divided by the temperature  $T$  to give

$$\frac{\omega}{T} = \left( \frac{mc^3}{Te^2} \right) \left( \frac{e^2}{J(m/T)c} \right)^3 \left[ 1 - \left( \frac{e^2}{J(m/T)c} \right)^2 \right]^{-1/2} \quad (19)$$

The left-hand side of this equation is a function of  $\omega/T$  while the right-hand side is a function of  $m/T$  (and  $e^2/c$ ). Therefore there is a unique functional relationship between the ratio  $\omega/T$  for the radiation and the ratio  $m/T$  for the matter. Thus the function  $J(m/T)$  of the matter can be reexpressed as a function of  $\omega/T$ , and visa versa. But then the relationship in Eq. (17) which connects the  $\sigma_{ltU-1}$ -scale-invariant ratio  $E_0/\omega^2$  of the radiation to the angular momentum  $J$  of the particle can be written to give  $E_0/\omega^2$  as a function of  $\omega/T$ . Again this is precisely the sort of connection which we expect to hold for a classical hydrogen atom in classical thermal radiation. And this connection can be made only in the case of the Coulomb potential which will arise in relativistic classical electron theory with classical electromagnetic zero-point radiation.

## IV. DISCUSSION

### A. Classical Statistical Mechanics

Traditional classical statistical mechanics involves equal probabilities on phase space and a limited total energy to be distributed over the phase space.[19] All temperatures are treated in exactly the same fashion, and there is no transition from a low-temperature to a high-temperature form of the theory. Such a theory can work satisfactorily for mechanical systems with their finite number of degrees of freedom. However, radiation with its infinite number of degrees of freedom does not allow such a treatment since the finite thermal energy will "leak out" to the divergent set of high-frequency modes. Quantum mechanics changes the rules of both electromagnetism and of statistical mechanics by introducing the idea of quanta. By contrast, classical electron theory with classical electromagnetic zero-point radiation keeps the rules of classical electromagnetism but makes a new choice for the homogeneous boundary condition of classical electromagnetism; this new choice invalidates the rules of traditional classical statistical mechanics because it introduces temperature-independent fluctuations which are present even at absolute zero. The presence of zero-point fluctuations, which are distinct from temperature-dependent fluctuations, means that

the theory now indeed involves a transition from a low-temperature to a high-temperature form.

In a classical theory of thermal equilibrium which includes zero-point radiation, each allowed system must have one parameter which indicates where the system is located in the collection of systems between zero-point energy and high-temperature energy. For a radiation normal mode of frequency  $\omega$ , the ratio  $\omega/T$  for the normal mode gives this location. The action variable  $J_\omega$  for the mode is a function of  $\omega/T$ . Thus if there is thermal radiation at temperature  $T$ , then the ratio  $\omega/T$  determines the energy  $U_\omega = J_\omega \omega$  of the normal mode of frequency  $\omega$ . If  $\omega/T \ll 1$ , the mode has its zero-point energy value  $U_\omega = (1/2)\hbar\omega$ , and if  $\omega/T \gg 1$ , then the mode has its high-temperature energy  $U_\omega = k_B T$ . Similarly, any charged particle of mass  $m$  in a Coulomb potential  $V(r) = -e^2/r$  must have a zero-point energy and a thermal energy above the zero-point value. The ratio  $m/T$  determines the location of the particle system along the continuum from zero-point to high-temperature value.

We note that nonrelativistic classical mechanical systems which depend upon several continuous parameters (such as a nonlinear oscillator of mass  $m$ , frequency  $\omega$ , and nonlinear parameter  $\Gamma$ ) expose ambiguous behavior in locating the system along the low-temperature versus high-temperature continuum. However, for *relativistic* classical electron theory with classical zero-point radiation, these multiparameter systems are not possible, and the scaling situation is enormously simplified, as is shown in the present article.

## B. Classical Electron Theory

Traditional classical electron theory was introduced by H. A. Lorentz during the last quarter of the nineteenth century.[1] The theory consisted of massive particles in nonrelativistic potentials which interacted with electromagnetic radiation through their fixed charge  $e$ . Lorentz assumed explicitly[20] that the homogeneous boundary condition on Maxwell's equations excluded fundamental electromagnetic radiation; rather all radiation arose from the acceleration of charged particles at a finite time. Traditional classical electron theory was able to account for a number of observed phenomena, including optical dispersion, Faraday rotation, and aspects of the normal Zeeman effect.[1][2] However, as is now reported in all the text books,[21] traditional classical electron theory was not able to account for the

observed Planck spectrum of blackbody radiation nor to solve the problem of collapse for Rutherford's atomic model. In the twentieth century, the idea of zero-point energy was introduced by Planck and extended to radiation by Nernst. However, it was not until the 1960s, beginning with Marshall's careful and extensive work,[22] that classical electron theory with classical electromagnetic zero-point radiation (under the title "stochastic electrodynamics") was shown to account for some phenomena which had previously been regarded as the exclusive domain of quantum physics, such as Casimir forces, van der Waals forces, diamagnetism, specific heats of solids,[4] and even the ground state of hydrogen.[5] The Planck spectrum of blackbody radiation held a paradoxical position. On the one hand, there were a number of derivations of the Planck spectrum from classical physics including zero-point radiation, but there were also several scattering calculations which suggested again that classical physics led inevitably to the divergent Rayleigh-Jeans spectrum for thermal radiation. The historical situation is probably best seen in a review[3] of classical electron theory with classical electromagnetic zero-point radiation written in 1975. In this review, one finds that the one change made away from Lorentz's classical electron theory is the introduction of classical electromagnetic zero-point radiation. There is no appreciation that the use of nonrelativistic mechanical systems in connection with Maxwell's relativistic electromagnetic theory is fundamentally inconsistent because the combination satisfies neither Galilean nor relativistic invariance. There is no awareness of the extreme restrictions which special relativity places on allowed interactions. The theory of 1975 which incorporates nonrelativistic potentials is valid in at most one inertial frame where the nonrelativistic approximation for the particle system is appropriate. However, the analysis given in the present work shows that stability of the thermal radiation spectrum within classical physics can be achieved only within a fully relativistic theory. In the present discussion, we have pointed out that *relativistic* classical electron theory with classical electromagnetic zero-point radiation has the  $\sigma_{\omega U^{-1}}$ -scaling behavior which will leave the classical zero-point radiation spectrum invariant under scattering by a classical hydrogen atom, and which will give the possibility of classical thermal radiation at a spectrum with finite thermal energy.



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